Supported by tetfund TERTIARY EDUCATION TRUST FUND

GOODNESS OF FIT TESTS FOR SOME GENERALIZED DISTRIBUTIONS

Ifeyinwa Vivian Omekam and Adebowale Olusola Adejumo*
Department of Statistics, University of Ilorin, Kwara State, Nigeria
*Corresponding author: aodejumo@unilorin.edu.ng

Received: December 20, 2019 **Accepted:** March 12, 2020

Abstract: Analysis of data without a pre-knowledge of the distribution that describes data may lead to misleading or irrelevant result. Distribution fitting to data often lead to the selection of the best fitting distribution for data analysis. In this article, some existing distributions are fitted to Maternal Mortality Ratio (MMR) data. Using the best fitting distribution obtained as base distribution, generalized distributions having additional parameters are then derived and subsequently fitted to MMR to assess goodness of fit. Generalized distributions improved goodness of fit.

Keywords: Goodness of fit; Lehmann alternatives, Marshall, Olkin methods

Introduction

Goodness of fit test is one of the most fundamental hypothesis testing problems (Lehmann and Romano, 2008). The extent to which a hypothesized probability distribution can describe a sample of data can be assessed using goodness of fit test. Therefore for accurate and relevant data analysis results, fitting generalized distributions to data before use in order to assess goodness of fit is as important as their derivation.

Tahir and Nadarajah (2015) in their study of generalized families of distributions discussed the exponentiated generalized families (Lehmann Alterntive 1 (LA1) and Lehmann Alternative 2 (LA2)) having additional parameter each. These generalized families are obtained from Lehmann Alternatives (Lehmann, 1953).

Marshall and Olkin (1997) presented the method of deriving generalized distribution by parameter induction into an existing distribution. Gupta and Kundu (2009) also discussed the Power Transformed Method (PTM), another method of parameter induction into an existing base distribution.

Let X be a base continuous random variable having the following functions;

Probability Density Function denoted by $f(x)$, Cumulative Distribution Function denoted by $F(x)$. If X is a lifetime random variable, then, the Survival Function is denoted by $s(x)$, the Hazard Function is denoted by $h(x)$, and the Reversed Hazard Function is denoted by $r(x)$.

For a generalized continuous random variable Y, obtained by introducing a parameter (c>0) to the distribution of the base random variable X, functions of Y obtainable from the generalized families presented by above authors are respectively summarized below;

Lehmann type I family

$$f_Y(x) = cf(x)F(x)^{c-1} \quad (1.1)$$

$$F_Y(x) = F(x)^c \quad (1.1a)$$

$$\bar{F}_Y(x) = 1 - F(x)^c \quad (1.1b)$$

$$h_Y(x) = cf(x)F(x)^{c-1}(1-F(x)^c)^{-1} \quad (1.1c)$$

$$r_Y(x) = cf(x)F(x)^{-1} \quad (1.1d)$$

Lehmann type II family

$$f_Y(x) = cf(x)[1-F(x)]^{c-1} = cf(x)[\bar{F}(x)]^{c-1} \quad (2.2)$$

$$F_Y(x) = 1-[1-F(x)]^c = 1-\bar{F}(x)^c \quad (2.2a)$$

$$\bar{F}_y(x) = [1-F(x)]^c = \bar{F}(x)^c \quad (2.2b)$$

$$h_Y(x) = cf(x)[1-F(x)]^{-1} = cf(x)[\bar{F}(x)]^{-1} \quad (2.2c)$$

$$r_Y(x) = cf(x)[1-F(x)]^{c-1}(1-[1-F(x)]^c)^{-1} \quad (2.2d)$$

Marshall-Olkin family

$$F_Y(x) = \frac{F(x)}{1-(1-c)[1-F(x)]} \quad (2.3)$$

$$f_Y(x) = \frac{cf(x)}{(1-(1-c)[1-F(x)])^2} \quad (2.3a)$$

$$\bar{F}_Y(x) = \frac{c\bar{F}(x)}{1-(1-c)[1-F(x)]} \quad (2.3b)$$

$$h_Y(x) = \frac{h(x)}{1-(1-c)[1-F(x)]} \quad (2.3c)$$

$$r_Y(x) = \frac{cr(x)}{1-(1-c)[1-F(x)]} \quad (2.3d)$$

Power transformed family

$$f_Y(x) = cx^{c-1}f(x^c) \quad (2.4)$$

$$F_Y(x) = F(x^c) \quad (2.4a)$$

$$\bar{F}_Y(x) = 1-F(x^c) = \bar{F}(x^c) \quad (2.4b)$$

$$h_Y(x) = cx^{c-1}h(x^c) \quad (2.4c)$$

$$r_Y(x) = cx^{c-1}r(x^c) \quad (2.4d)$$

Omekam and Adejumo (2017) generated families of generalized distributions by sequentially applying methods in permutations of five distinct parameter induction (including those from families mentioned above) taken two methods at a time. Among other generalized distributions derived by authors are those that are found in Permutations 4 and 10 which may respectively be called Lehmann Type II-Power Transformed Family and Lehmann Type I-Power Transformed Family. Functions of these two generalized families with two additional parameters are represented below

Given a base random variable X, let Y be a generalized continuous random variable obtained by introducing ashape parameter (c>0) to the distribution of the base random variable X, and let Z be another continuous variable belonging to a family of generalized distributions obtained by introducing another parameter(t>0) to Y.

Lehmann type II-power transformed family

$$f_Z(x) = ctx^{c-1} f(x^c)[\bar{F}(x^c)]^{t-1} \quad (2.3)$$

$$F_Z(x) = 1 - [\bar{F}(x^c)]^t \quad (2.3a)$$

$$\bar{F}_Z(x) = [\bar{F}(x^c)]^t \quad (2.3b)$$

$$h_Z(x) = ctx^{c-1} f(x^c)[\bar{F}(x^c)]^{-1} \quad (2.3c)$$

$$r_Z(x) = ctx^{c-1} f(x^c)(1-F(x^c))^{t-1}\left(1-(1-F(x^c))^t\right)^{-1}$$

$$= ctx^{c-1} f(x^c)[\bar{F}(x^c)]^{t-1}(1-[\bar{F}(x^c)]^t)^{-1} \quad (2.3d)$$

Lehmann type I-power transformed family

$$f_Z(x) = ctx^{c-1} f(x^c)[F(x^c)]^{t-1} \quad (2.4)$$

$$F_Z(x) = [F(x^c)]^t, \quad (2.4a)$$

$$\bar{F}_Z(x) = 1 - [F(x^c)]^t \quad (2.4b)$$

$$h_Z(x) = ctx^{c-1} f(x^c)[F(x^c)]^{t-1}\left(1-[F(x^c)]^t\right)^{-1} \quad (2.4c)$$

$$r_Z(x) = ctx^{c-1} f(x^c)[F(x^c)]^{-1} = ctx^{c-1} r(x^c) \quad (2.4d)$$

Nadarajah and Kotz (2003) introduced Exponentiated Fretchet distribution having three parameters that generalized the two-parameter Frechet distribution. Cordeiro *et al*. (2013) proposed a method of introducing two parameters to a continuous distribution and thereafter obtained some models from same method. Among the models obtained is the Exponentiated Generalized Frechet (EGF). Four of the special models obtained were fitted to four different real datasets and compared with those of three other sub-models. The results were in favour of proposed models based on Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and likelihood ratio test.

Krishna *et al*. (2013) introduced the Marshall-Olkin Extended Frechet distribution (MOEFR) with an additional parameter generalizing the two-parameter Frechet distribution. MOEFR distribution was compared with two-pararmeter Frechet distribution using a data set consisting of 72 observations of survival times of guinea pigs injected with different doses of tubercle bacilli. The good fit provided by MOEFR based on Kolmogorov-Smirnov (K-S) distances and associated P-values, AIC, and BIC informed its recommendation as a competitive model to the Frechet distribution.

Materials and Methods

MMR is a measure of maternal mortality that measures maternal mortality as the number of maternal deaths per 100,000 live births.

Data collection

Data on the number of maternal death and live birth were collected for fifty one countries listed in the appendix which includes twenty one developing countries for a period of eleven years (2004 to 2014). Country inclusion criterion was civil registration data characterization by Maternal Mortality Estimation Inter Agency Group (MMEIG). The goal of using the country inclusion criterion is to avoid using data with wide range of uncertainty. Forty two out of the fifty one countries included were countries whose civil registration data were characterized as complete, with good attribution of cause of death while seven were countries classified as lacking good complete registration data but where registration or other types of data are available. The remaining two countries (Cyprus and Malaysia) had no nationally representative data on maternal death but their live birth data for the years considered were complete. MMEIG estimates were used for the two countries. The study would have used a larger sample size but sample size was a function of criterion stated. Secondary data was used and the sources are given below:

Number of maternal death: World Development Indicators (WDI) Number of live birth: Eurostat and United Nations Statistics Division (UNSD)

Distribution fitting and generalization of distribution

The shape of MMR data was ascertained using histogram plot. The shape revealed in the plot suggested plausible existing distributions that were then fitted to data. K-S goodness of fit test was used in determining the goodness of fit of plausible distributions and AIC was subsequently used to select the best model. The best fitted model was generalized to produce six generalized distributions. Goodness of fit test was then carried out for the four generalized distributions with one additional parameter and two generalized distributions with two additional parameters obtained from generalizing the best fitted model. Generalized distribution for MMR was finally selected from these six generalized distributions using K-S distances and AIC. Easy Fit and R software were employed.

Results and Discussion

Let the continuous random variable, MMR, be represented by X. The functions of the distribution that describes X are given as follows:

Probability Density Function denoted by $f(x)$, Cumulative Distribution Function denoted by $F(x)$. If X is a lifetime random variable, then, the Survival Function is denoted by $s(x)$, the Hazard Function is denoted by $h(x)$, and the Reversed Hazard Function is denoted by $r(x)$.

Let Y be a generalized continuous random variable obtained by introducing a shape parameter (c>0) to the distribution of the base random variable X, and let Z be another continuous variable belonging to a family of generalized distributions with two additional shape parameters($c > 0$ and $t>0$) also obtained from the distribution of X.

The shape of MMR data is deduced from the histogram for X which will then suggest plausible existing distributions that will be subsequently fitted to data. Generalized distributions obtained for the best fitted existing distribution are then fitted to X. X is concluded to follow the generalized distribution with the best fit to data.

Distribution fitting of some existing distributions to data

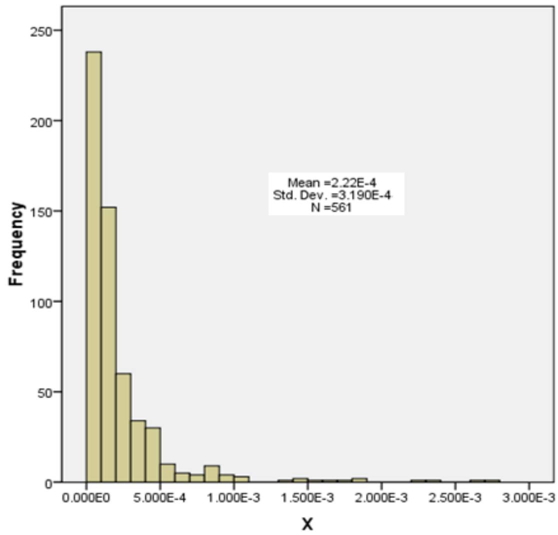

Fig. 1: Histogram for MMR

Histogram shows a positive skew and unimodal shape.

Table 1: Goodness of fit of some existing distributions to data

Distribution	Kolmogorov Smirnov			AIC		
	Statistic	P-value	Rank	Statistic	Rank	Relative Likelihood
Beta	0.15969	0.00000				
Burr	0.05183	0.09483	3	-8546.016	3	0.00021029
Dagum	0.04357	0.23041	1	-8558.09	2	0.08803683
Exponential	0.14186	0.00000				
Fatigue Life	0.12951	0.00000				
Frechet	0.05164	0.09693	2	-8562.95	1	
Gamma	0.29941	0.00000				
Gen. Gamma	0.19235	0.00000				
Inverse Gaussian	0.10194	0.00000				
Pareto 2	0.16234	0.00000				
Rayleigh	0.33499	0.00000				
Rice	0.50072	0.00000				
Weibull	0.15485	0.00000				

The best fitted distribution based on K-S statistics and associated P-values is the three parameter Dagum distribution with a P-value of 0.23041 while the two-parameter Frechet distribution is selected by AIC. The relative likelihood of Dagum distribution indicates that Dagum is 0.08803683 probable as the Frechet to minimize information loss, therefore, giving a strong evidence for Frechet distribution as the best distribution for X.

Generalization of the best fitted distribution

The Frechet distribution

$$f(x)=\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right) \tag{3.1}$$

$$F(x)=\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right) \tag{3.2}$$

$$\overline{F}(x)=1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right) \tag{3.3}$$

$$h(x)=\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\left[1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\right]^{-1} \tag{3.4}$$

$$r(x)=\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1} \tag{3.5}$$

α, β > 0, ϰ > 0. α and β are shape and scale parameters, respectively

Generalized Frechet distributions

1. Lehmann Type I Frechet Distribution (LIFD)

Substituting $f(x)$ in (3.1) for $f(x)$ and $F(x)$ in (3.2) for $F(x)$ in (1.1)

$$f_y(x)=c\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c-1}$$

$$=c\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c} \tag{3.6}$$

Substituting $F(x)$ in (3.2) for $F(x)$ in (1.1a)

$$F_y(x)=\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c} \tag{3.7}$$

Substituting $\overline{F}(x)$ in (3.2) for $F(x)$ in (1.1b)

$$\overline{F}_y(x)=1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c} \tag{3.8}$$

Substituting $f(x)$ in (3.1) for $f(x)$ and $F(x)$ in (3.2) for $F(x)$ in (1.1c)

$$h_y(x)=c\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c-1}\left(1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c}\right)^{-1}$$

$$=c\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c}\left(1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c}\right)^{-1} \tag{3.9}$$

Substituting $f(x)$ in (3.1) for $f(x)$ and $F(x)$ in (3.2) for $F(x)$ in (1.1d)

$$r_y(x)=c\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{-1}$$

$$=c\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1} \tag{3.10}$$

2. Lehmann Type II Frechet Distribution (LIIFD)

Substituting $f(x)$ in (3.1) for $f(x)$ and $\overline{F}(x)$ in (3.3) for $\overline{F}(x)$ in (2.2)

$$f_y(x)=c\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\left[1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\right]^{c-1} \tag{3.11}$$

Substituting $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.2a)

$$F_y(x)=1-\left(1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\right)^{c} \tag{3.12}$$

Substituting $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.2b)

$$\bar{F}_y(x)=\left[1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\right]^{c} \tag{3.13}$$

Substituting $f(x)$ in (3.1) for $f(x)$ and $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.2c)

$$h_y(x)=c\frac{\alpha}{\beta}^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\left[1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\right]^{-1} \tag{3.14}$$

Substituting $f(x)$ in (3.1) for $f(x)$ and $F(x)$ in (3.2) for $F(x)$ in (2.2d)

$$r_y(x)=c\frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\left[1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\right]^{c-1}\left(1-\left[1-\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)^{c}\right]\right)^{-1} \tag{3.15}$$

3. Marshall-Olkin Frechet Distribution (MOFRD)

Substituting $F(x)$ in (3.2) for $F(x)$ in (2.3)

$$F_y(x)=\frac{\exp(-(\beta/x)^{\alpha})}{1-(1-c)[1-\exp(-(\beta/x)^{\alpha})]} \tag{3.16}$$

Substituting $f(x)$ in (3.1) for $f(x)$ and $F(x)$ in (3.2) for $F(x)$ in (2.3a)

$$f_y(x)=\frac{c\alpha\beta^{-1}(\beta/x)^{\alpha+1}\exp(-(\beta/x)^{\alpha})}{(1-(1-c)[1-\exp(-(\beta/x)^{\alpha})])^{2}} \tag{3.17}$$

Substituting $F(x)$ in (3.2) for $F(x)$ and $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.2b)

$$\bar{F}_y(x)=\frac{c[1-\exp(-(\beta/x)^{\alpha})]}{1-(1-c)[1-\exp(-(\beta/x)^{\alpha})]} \tag{3.18}$$

Substituting $h(x)$ in (3.4) for $h(x)$ and $F(x)$ in (3.2) for $F(x)$ in (2.3c)

$$h_y(x)=\frac{\alpha\beta^{-1}(\beta/x)^{\alpha+1}\exp(-(\beta/x)^{\alpha})[1-\exp(-(\beta/x)^{\alpha})]^{-1}}{1-(1-c)[1-\exp(-(\beta/x)^{\alpha})]} \tag{3.19}$$

Substituting $r(x)$ in (3.5) for $r(x)$ and $F(x)$ in (3.2) for $F(x)$ in (2.3d)

$$r_y(x)=\frac{c\alpha\beta^{-1}(\beta/x)^{\alpha+1}}{1-(1-c)[1-\exp(-(\beta/x)^{\alpha})]} \tag{3.20}$$

4. Power Transformed Frechet Distribution (PTFD)

Substituting $f(x)$ in (3.1) for $f(x)$ in (2.4)

$$f_y(x)=cx^{c-1}\frac{\alpha}{\beta}\left(\frac{\beta}{x^{c}}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x^{c}}\right)^{\alpha}\right) \tag{3.21}$$

Substituting $F(x)$ in (3.2) for $F(x)$ in (2.4a)

$$F_y(x)=\exp\left(-\left(\frac{\beta}{x^{c}}\right)^{\alpha}\right) \tag{3.22}$$

Substituting $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.4b)

$$\bar{F}_y(x) = 1 - \exp\left(-\left(\frac{\beta}{x^c}\right)^{\alpha}\right) \tag{3.23}$$

Substituting $h(x)$ in (3.4) for $h(x)$ in (2.4c)

$$h_y(x) = cx^{c-1}\frac{\alpha}{\beta}\left(\frac{\beta}{x^c}\right)^{\alpha+1}\exp\left(-\left(\frac{\beta}{x^c}\right)^{\alpha}\right)\left[1-\exp\left(-\left(\frac{\beta}{x^c}\right)^{\alpha}\right)\right]^{-1} \tag{3.24}$$

Substituting $r(x)$ in (3.5) for $r(x)$ in (2.4d)

$$r_y(x) = cx^{c-1}\frac{\alpha}{\beta}\left(\frac{\beta}{x^c}\right)^{\alpha+1} \tag{3.25}$$

5. Lehmann Type II Power Transformed Frechet Distribution (LIIPTFD)

Substituting $f(x)$ in (3.1) for $f(x)$ and $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.3)

$$f_z(x) = ctx^{c-1}\alpha\beta^{-1}(\beta/x^c)^{\alpha+1}\exp(-(\beta/x^c)^{\alpha})[1-\exp(-(\beta/x^c)^{\alpha})]^{t-1} \tag{3.26}$$

Substituting $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.3a)

$$F_z(x) = 1-[1-\exp(-(\beta/x^c)^{\alpha})]^t \tag{3.27}$$

Substituting $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.3b)

$$\bar{F}_z(x) = [1-\exp(-(\beta/x^c)^{\alpha})]^t \tag{3.28}$$

Substituting $f(x)$ in (3.1) for $f(x)$ and $\bar{F}(x)$ in (3.3) for $\bar{F}(x)$ in (2.3c)

$$h_z(x) = ctx^{c-1}\alpha\beta^{-1}(\beta/x^c)^{\alpha+1}\exp(-(\beta/x^c)^{\alpha})[1-\exp(-(\beta/x^c)^{\alpha})]^{-1} \tag{3.29}$$

Substituting $f(x)$ in (4.1) for $f(x)$ and $\bar{F}(x)$ in (4.3) for $\bar{F}(x)$ in (2.3d)

$$r_z(x) = ctx^{c-1}\alpha\beta^{-1}(\beta/x^c)^{\alpha+1}\exp(-(\beta/x^c)^{\alpha})[1-\exp(-(\beta/x^c)^{\alpha})]^{t-1}(1-[1-\exp(-(\beta/x^c)^{\alpha})]^{t-1} \tag{3.30}$$

6. Lehmann Type I-Power Transformed Frechet Distribution (LIPTFD)

Substituting $f(x)$ in (3.1) for $f(x)$ and $F(x)$ in (3.2) for $F(x)$ in (2.4)

$$f_z(x) = ctx^{c-1}\alpha\beta^{-1}(\beta/x^c)^{\alpha+1}\exp(-(\beta/x^c)^{\alpha})[\exp(-(\beta/x^c)^{\alpha})]^{t-1} \tag{3.31}$$

Substituting $F(x)$ in (3.2) for $F(x)$ in (2.4a)

$$F_z(x) = \exp(-(\beta/x^c)^{\alpha})^t \tag{3.32}$$

Substituting $F(x)$ in (3.2) for $F(x)$ in (2.4b)

$$\bar{F}_z(x) = 1-\exp(-(\beta/x^c)^{\alpha})^t \tag{3.33}$$

Substituting $f(x)$ in (3.1) for $f(x)$ and $F(x)$ in (3.2) for $F(x)$ in (2.4c)

$$h_z(x) = ctx^{c-1}\alpha\beta^{-1}(\beta/x^c)^{\alpha+1}\exp(-(\beta/x^c)^{\alpha})^t[1-\exp(-(\beta/x^c)^{\alpha})^t]^{-1} \tag{3.34}$$

Substituting $r(x)$ in (3.5) for $r(x)$ in (2.4d)

$$r_z(x) = ctx^{c-1}\alpha\beta^{-1}(\beta/x^c)^{\alpha+1} \tag{3.35}$$

Distribution fitting of generalized distributions to data

Table 2: Goodness of fit of generalized distributions to data

Distribution	Kolmogorov Smirnov			AIC		
	Statistic	P-value	Rank	Statistic	Rank	Relative Likelihood
Frechet	0.05164	0.09693	4	-8562.95	2	0.994018
LIIFD	0.04726	0.1598379	1	-8561.582	3	0.501576
MOFRD	0.04764	0.1534482	3	-8562.962	1	
PTFD	0.04741	0.1572908	2	-8561.572	4	
LIIPTFD	0.04726	0.1598379	1	-8559.582	5	
LIPTFD	0.04741	0.1572908	2	-8559.572	6	
LIFD	0.04741	0.1572908	2	-8561.572	4	

The best fitted distribution based on K-S statistics and associated P-values is Lehmann Type II distribution with a P-value of 0.1598379 while Marshall-Olkin Frechet distribution is selected by AIC. The relative likelihood of Frechet distribution indicates that Frechet distribution is 0.994018 probable as MOFRD distribution to minimize information loss, therefore, MOFRD is selected as an alternative reference distribution to Frechet distribution for X.

Conclusion

Krishna *et al.* (2013) applied MOEFR to one data set and found it to be a competitive model to the Frechet. Cordeiro *et al*. (2013) applied special generalized models derived to different data sets andresults obtained revealed their suitability to data sets. The relative likelihood of the Frechet distribution obtained in this study led to the choice of MOFRD as an alternative reference distribution to the Frechet distribution for MMR. These results provide evidence for Marshall-Olkin Frechet distribution to perform as well as and even much better than the Frechet distribution. In general, results give evidence that generalized distribution often improve goodness of fit. More properties of derived generalized distributions need to be investigated and used in subsequent analyses.

Conflict of Interest

Authors declare there is no conflict of interest related to this study.

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APPENDIX

List of Countries

Austria, Azerbaijan, Belgium, Bulgaria, Bosnia & Herz., Belarus, Switzerland, Costa Rica, Cuba, Cyprus, Czech Republic, Germany, Denmark, Egypt Arab Rep., Spain, Estonia, Finland, France, United Kingdom, Greece, Hungary, Ireland, Israel, Italy, Japan, Kyrgyz Republic, Korea, Rep., Lithuania, Luxembourg, Latvia, Moldova, Maldives, Macedonia FYR, Montenegro, Mauritius, Malaysia, Netherlands, Norway, New Zealand, Poland, Portugal, Romania, Russian Fed., Singapore, Serbia, Suriname, Slovenia, Sweden, Turkey, Ukraine, United States.